Abstraction and idealization in geometry and topology

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Outline

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Perspective space

- Both Euclidean geometry and physics (or at least some of its fragments) aspire to describe the most general properties of something that, after Bertrand Russell, can be called *the perspective space*.
- For the sake of the goals of the talk it will be enough to understand the perspective space (or simply *the space*) as the sensually accessible world that surrounds us.

Perspective space

- The space has three dimensions (we will not be taking time into account), that is, intuitively, everything can be measured in three basic ways.
- Bodies are parts of the space and all of them have three-dimensions as well—in every day experience we do not encounter anything like less than three-dimensional entities.

Outline

Geometry

- Geometry may be viewed (from at least one stance) as a mathematical theory of most general properties of the space.
- The impulse for its development came from measuring bodies and comparing spatial position of things in the perspective space.
- The inherent feature of parts of the space are their imperfections: straight lines are never straight (and are never lines!), planes are not flat, spheres are never perfect.
- The first key factor: idealization.

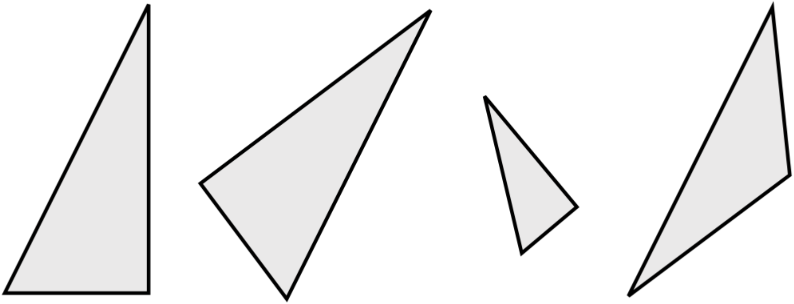
Idealization in geometry

- Idealization may be understood as introduction of «perfect» objects.
- Points, straight lines, planes, spheres.
- Idealization is a cognitive process.

Abstraction in geometry

- The second key factor: abstraction.
- «Identification» of objects according to some distinguished shared property.
- Once we have ideal objects we can abstract their features.
- Abstraction is a mathematical operation.

Abstraction in geometry – congruence



Abstraction in geometry – similarity

Idealization or abstraction?

Definition

A point is that which has position but not dimensions.

Definition

A line is length without breadth.

Definition

A surface is that which has length and breadth [but not depth (or thickness)].

Outline

Set theoretical approach to point-based geometry

In Foundations of Geometry by K. Borsuk and W. Szmielew, with reference to David Hilbert's book of the same title, the authors examine structures of the form $\langle \mathbf{P}, \mathfrak{L}, \mathfrak{P}, \mathbf{B}, \mathbf{D} \rangle$, in which:

- $\mathbf{P}$ is a non-empty set of points,
- $\mathfrak{L}$ and $\mathfrak{P}$ are subsets of $\mathcal{P}(\mathbf{P})$,
- $\mathbf{B}$ and $\mathbf{D}$ are, respectively, ternary and quaternary relation in $\mathbf{P}$.
- Elements of $\mathfrak{L}$ and $\mathfrak{P}$ are called, respectively, lines and planes, $\mathbf{B}$ is called betweenness relation and $\mathbf{D}$ equidistance relation.
- We put specific axioms on $\mathbf{P}$, $\mathfrak{L}$, $\mathfrak{P}$, $\mathbf{B}$ and $\mathbf{D}$, and in this way we obtain a system of geometry that would probably satisfy Euclid and his contemporaries.

Incidence relation

- Sometimes an additional relation in $\mathbf{P} \times \mathfrak{L}$ and $\mathbf{P} \times \mathfrak{P}$ are introduced, the so called incidence relations, in our case will be denoted by 'ϵ'.
- In case p is a point and L is a line we read '$p \mathrel{\epsilon} L$' as p is incident with L (similarly for planes)

Set theoretical approach to point-based geometry

We can modify the above approach to start with structures $\langle \mathbf{P}, \mathbf{B}, \mathbf{D} \rangle$ and subsequently take such a collection of axioms that $\mathfrak{L}$ and $\mathfrak{P}$ will be definable be means of $\mathbf{B}$.
The set of lines can be defined in the following way

$$X \in \mathfrak{L} \stackrel{\mathrm{df}}{\longleftrightarrow} \exists_{p,q \in \mathbf{P}} (p \neq q \wedge$$
$$X = \{r \in \mathbf{P} \mid \langle r, p, q \rangle \in \mathbf{B} \vee \langle p, r, q \rangle \in \mathbf{B} \vee \langle p, q, r \rangle \in \mathbf{B}\} \cup \{p, q\}),$$

where the condition '$\langle r, p, q \rangle \in \mathbf{B}$' says that point p is between points q and r.

Set theoretical approach to point-based geometry

To define $\mathfrak{P}$, first we introduce a new relation $\mathbf{L} \subseteq \mathbf{P}^3$, so called relation of collinearity of points

$$\langle p, q, r\rangle \in \mathbf{L} \stackrel{\mathrm{df}}{\longleftrightarrow} \exists_{X \in \mathfrak{L}}(p \in X \wedge q \in X \wedge r \in X).$$

Subsequently we define a triangle, whose cones are located in three points p, q, r (in symbols '$\mathrm{tr}(pqr)$') that are not collinear

$$\begin{aligned} &\neg \mathbf{L}(p, q, r) \longrightarrow \\ &\qquad \mathrm{tr}(pqr) := \{a \in \mathbf{P} \mid a = p \vee a = q \vee a = r \vee \\ &\qquad\qquad \langle p, a, q\rangle \in \mathbf{B} \vee \langle p, a, r\rangle \in \mathbf{B} \vee \langle q, a, r\rangle \in \mathbf{B}\}.\end{aligned}$$

Now we define a plane

$$\begin{aligned} &X \in \mathfrak{P} \stackrel{\mathrm{df}}{\longleftrightarrow} \exists_{p,q,r \in \mathbf{P}}\Big[\neg \mathbf{L}(p, q, r) \wedge X = \Big\{c \in \mathbf{P} \mid \\ &\quad \exists_{a,b \in \mathbf{P}}[a \neq b \wedge a, b \in \mathrm{tr}(pqr) \wedge \langle c, a, b\rangle \in \mathbf{B} \vee \langle a, c, b\rangle \in \mathbf{B}]\Big\}\Big].\end{aligned}$$

Set theoretical approach to point-based geometry

Mario Pieri (1860-1913)

- La geometria elementare istituita sulle nozioni “punto” é “sfera”, *Matematica e di Fisica della Società Italiana delle Scienze*, **vol. 15**, 1908, 345–450.
- In Polish: *Geometrja elementarna oparta na pojęciach „punktu” i „sfery”*, Gebether i Wolff, Warsaw, 1915.

Set theoretical approach to point-based geometry

- It was proven by Pieri that to construct a system of Euclidean geometry one actually needs only two primitive notions: that of point and that of equidistance relation, which in the Pieri's system case is a ternary relation among points.
- Denoting this relation by means of '$\triangle$' we can say that while doing geometry in Pieri's manner we analyze elementary structures $\langle \mathbf{P}, \triangle \rangle$, where $\triangle \subseteq \mathbf{P}^3$.
- Now we of course have to choose axioms to define $\mathfrak{L}$, $\mathfrak{P}$, **B** and **D** in such a way to be able to prove that this approach is definitionally equivalent to Hilbert's one system.

Set theoretical approach to point-based geometry

Thus, in light of the above constructions, we conclude that to construct Euclidean geometry:

- one can do with five primitive notions: of point, of line, of plain, of betweenness relation and of equidistance relation
- one can do with three primitive notions: of point, of betweenness relation and of equidistance relation
- one can do with two primitive notions: of point, of betweenness relation and of triangle relation
- the notion of point is the only one that is present in every approach.

Type-theoretical perspective

From point of view of type theory we have the following situations:

- if points, lines and planes are assumed as primitives, then they have the type ι of individuals
- the incidence relation has the type (ι, ι), betweenness (ι, ι, ι) and equidistance $(\iota, \iota, \iota, \iota)$

* if points only are assumed as primitives, then they have the type ι of individuals
* lines and planes explained in terms of points have the type (ι)
* incidence relation has the type $(\iota, (\iota))$
* betweenness still has (ι, ι, ι) and equidistance $(\iota, \iota, \iota, \iota)$.

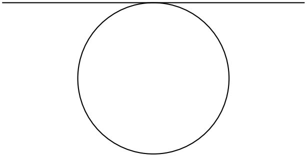
Outline

Back to antiquity

It was probably Pythagoreans who first formulated abstract geometrical notions:

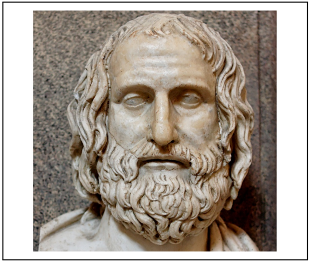
- of a line as an object without thickness (width)
- of a point as an object without any dimensions
- of a circle as a line whose all points are equidistant from a given point
- of a line tangent to a circle as a line which shares exactly one point with a circle

Objections against classical point-based geometry



Protagoras (c. 490-420 BC)

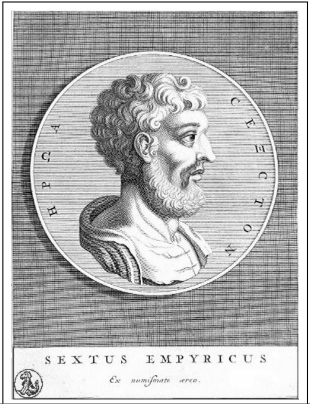
- a tangent shares a segment with a circle
- geometry is absurd since it deals with non-existent objects



Objections against classical point-based geometry

The main objection against points: they are treated as an ultimate constituent of reality, while we do not experience any objects that bear any resemblance to them.

Sextus Empiricus (c. 160-210 AD)
Against the Geometers in: Against the Professors



Objections against classical point-based geometry

- objections against geometry as dealing with abstract objects
- objections against geometry as postulating existence of objects that cannot be confirmed empirically, i.e. idealized objects

Objections against classical point-based geometry

- Points, from which in a geometrical or a physical model space is built, are neither sensually experienced nor its existence can be derived from data (both by some experiment or some kind of reasoning); moreover we cannot point to objects in the real world, that could be «natural» counterparts of points.
- The space of geometry and its «parts» as Cantorian sets are abstract and as such they cannot be experienced empirically; the perspective space and its parts are concrete (sensually experienced).
- All objects that exist in the perspective space have dimensions and parts, so points cannot be elements of this space.

Objections against classical point-based geometry

- The problems described above were a stimulus to search for some other, different from point-based one, approach to geometry. Those approaches are usually named region-based, point-free or pointless.
- Those geometries do not either aim at replacing classical geometry with some other formal science or question usefulness of the notion of point. The introduction of this notion to science by the ancients was ingenious and enabled really impressive development of both mathematics and physics.

Outline

Scenting in the 'classes' of Whitehead and Russell and in the 'extensions of concepts' of Frege, the aroma of mythical specimen from a rich gallery of invented objects, I am unable to rid myself of an inclination to sympathize 'on credit' with the authors' doubts as to whether such 'such classes', do exist in the world.

S. Leśniewski

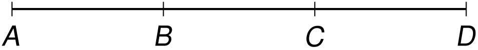
Go there!

Leśniewski's assumptions about sets

1. If x is a set of φ-ers, then there is at least one φ-er.
2. if there is only one object x which is an φ-er, then x is identical with the set of φ-ers
3. x is element of a set S iff for some condition φ, S is a set of φ-ers and x is an φ-er.

Leśniewski's assumptions about sets

4. It is very common that one and the same object is identical with various sets of objects. For example, consider the line segment *AD* below. Then *AD* is identical with the sets which consist of (e.g.):
   - *AB* and *BD*,
   - *AB*, *BC* and *CD*.



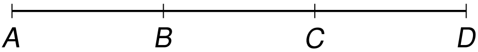
Leśniewski's assumptions about sets

Let $[\![y \mid y \text{ is an } \varphi\text{-er}]\!]$ be the class of φ-ers. With this at hand we may reformulate the above points as follows:

1′. if there exists $[\![y \mid y \text{ is an } \varphi\text{-er}]\!]$, then $[\![y \mid y \text{ is an } \varphi\text{-er}]\!]$ has at least one element
2′. if $[\![y \mid y \text{ is an } \varphi\text{-er}]\!]$ has exactly one element a, then $a = [\![y \mid y \text{ is an } \varphi\text{-er}]\!]$, that is $a = [\![a]\!]$
3′. $x \sqsubset S$ iff for some condition φ, $S = [\![y \mid y \text{ is an } \varphi\text{-er}]\!]$ and x is an φ-er.

Leśniewski's assumptions about sets

4′. What is stated in point 4 above in reference to the figure below may be expressed in the notation as follows:
   - $AD = [\![AB, BD]\!]$
   - $AD = [\![AB, BC, CD]\!]$

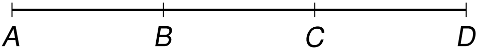


Cantorian set theory

1C. The set $\{x \in \mathbf{N} \mid x \neq 0 \wedge x + x = x\}$ has no elements.

2C. $\{\emptyset\} \neq \emptyset$.

4C. $AD \neq \{AB, BD\}$ and $AD \neq \{AB, BC, CD\}$, since treated as a Cantorian set the segment AD may only be considered as the set of points incident with it (but not the set of segments).



Leśniewski's nominalism

Leśniewski did not recognize existence of abstract objects, like Cantorian sets for example. From ontological point of view mereology as theory of fusions is better for nominalism. The main reasons for this are:

- first, in the process of joining objects to form fusions ontological status of fusion may be inherited from that of its constituents, thus if we fuse concrete objects what we obtain may be a concrete object; this is different from Cantorian sets which are always abstract entities;
- second, from nominalistic point of view it is natural to talk about parts of objects, while set theoretical $\in$ does not concern any relationship between concrete objects.

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Leśniewski's nominalism – mereology

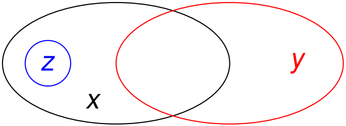
- The idea was to found the notion of set on the notion of parthood.
- The key notion of mereology is that of mereological sum or fusion, which may be considered nominalistic interpretation of the notion of set.

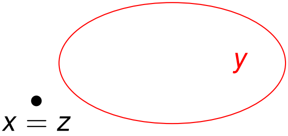
Leśniewski's nominalism – mereology

- Every object is part of itself (reflexivity).
- If x is part of y and y is part of x, then $x = y$ (antisymmetry).
- If x is part of y and y is part of z, then x is part of z (transitivity).

Leśniewski's nominalism – mereology

- If x is not part of y, then there is z which is part of x and is external to y, that is x and y does not have a common part (Strong Supplementation Principle or Polarization Condition).





Leśniewski's nominalism – mereology

Definition (Mereological sum (fusion))

An object x is a mereological fusion of all elements of φ-ers iff every φ-er is part of x and every part of x overlaps some element φ-er.

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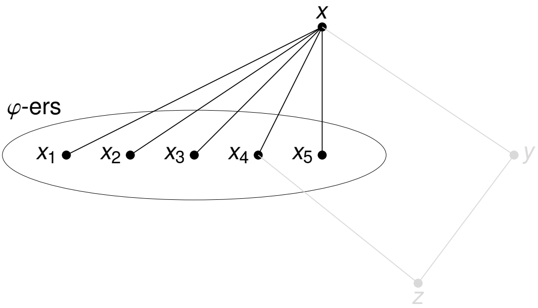

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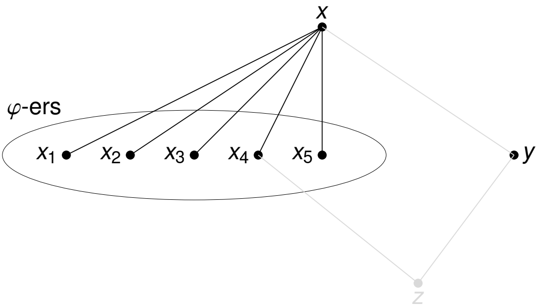

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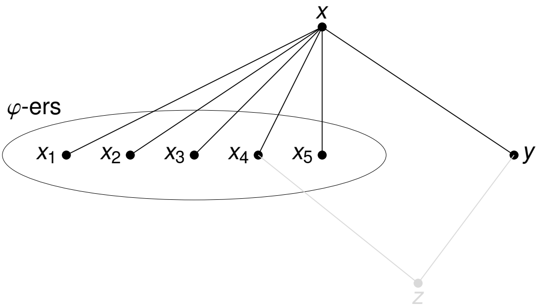

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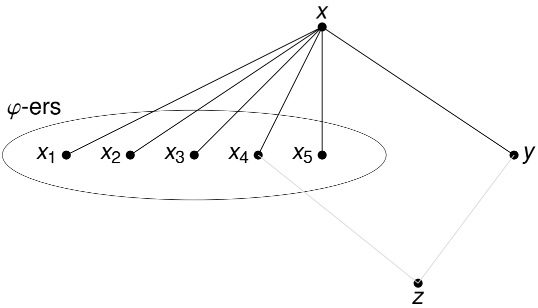

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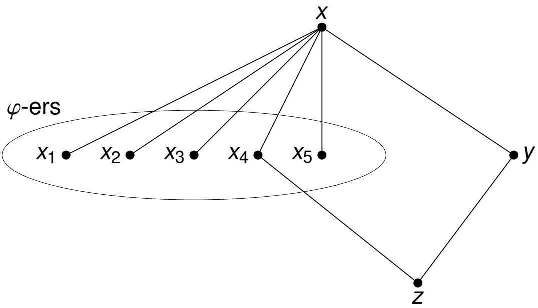

Leśniewski's nominalism – mereology

Axioms of existence of mereological sums:

- for every x and y there exists the mereological sum of x and y
- every group of objects has its mereological sum
- Leśniewski adopted the latter (caveat!)

Definition

By mereology I will understand a partial order which satisfies polarization condition plus the weak existence axiom. In case the strong existence axiom I will talk about complete mereology.

Outline

Leśniewski's views on geometry

- It was Leśniewski himself who formulated one of the first approaches to geometry in which space and figures are not Cantorian sets.
- He assumed however that among figures there are less than three-dimensional objects and that dimensionless points are parts of space.
- Segments, lines, planes and other figures were, at the same time, parts of space and mereological sets of points.
- Space itself was the mereological sum (fusion) of all points. Therefore his approach to geometry was still point-based.
- The difference between his system and classical geometry lied in the fact that he used mereological tools instead of set theoretical ones.

Leśniewski's views on geometry

From a formal point of view Leśniewski's approach to geometry can be characterized as follows.
Assume that:

- $\mathbf{s}$ is space,
- $\sqsubset$ is a relation of *being a proper part* and
- $\mathbf{Pt}$, F, L and P are, respectively, Cantorian sets of all points, figures, lines and planes.

Leśniewski's views on geometry

Then we have that

1. (i) $\mathbf{s} \neq \mathbf{Pt}$ (space is not the set of all points);
2. (ii) $\mathbf{s}$ is the mereological sum of all points;
3. (iii) $\mathbf{s} \in F$ (space is one of figures);
4. (iv) $x \in F$ and $x \neq \mathbf{s}$ iff $x \sqsubset \mathbf{s}$ (every figure which is different from space is its part and conversely, every part of space is a figure);
5. (v) $\mathbf{Pt}, L, P \subseteq F$ (all points, lines and planes are figures, therefore they are parts of space).

Leśniewski's views on geometry

From an ontological point of view Leśniewski's approach has actually the same faults as classical point-based geometries. To tell the truth neither space nor figures are any longer identified with Cantorian sets of points, but still space is «infested with» less than three-dimensional objects whose counterparts are not present in the perspective space.

Outline

Russell on points

Our Knowledge of the External World

> *It is customary to think of points as simple and infinitely small, but geometry in no way demands that we should think of them in this way. All that is necessary for geometry is that they should have mutual relations possessing certain enumerated abstract properties, and it may be that an assemblage of data of sensation will serve this purpose.*

The task of region-based geometry

It thus can be said that the task of region-based geometry is to construct such mathematical objects among which there hold the same relations as among «ordinary» points and which fulfill the following requirements

- their ontological status will be less problematic than in case of Euclidean points;
- its «building material», out of which they will be constructed, could be naturally and intuitively interpreted in the perspective space.

Ontological commitments of region-based geometry

- Instead of the set of points we have the set of objects that are called solids, regions or spatial bodies. Let **R** be the set of all regions.
- **R** is ordered by the part of relation whose satisfies the axioms of the (complete) mereology.
- The space **s** (if assumed to exists) is usually the unity of **R**.
- Lines and planes are not elements of **R**. Intuitively, **R** contains only three-dimensional and «regular» parts of space.

Points as Cantorian sets of regions

- Points are either Cantorian sets of regions or Cantorian sets of sets of regions. Let Π be the set of all points. Then:

$$\Pi \subseteq \mathcal{P}(\mathbf{R}) \qquad \text{or} \qquad \Pi \subseteq \mathcal{P}(\mathcal{P}(\mathbf{R}))\,.$$

- $\Pi \neq \mathbf{s}$ (the set of all points is not the space).
- Observe the impact of low level of idealization on the level of abstraction.

Figures as sets of points

- A figure is defined in a standard way, as a nonempty set of points:
$$\mathfrak{F} \coloneqq \mathcal{P}_{+}(\Pi)\,.$$
- The set of all points is a figure: $\Pi \in \mathfrak{F}$.
- But:
$$\Pi \cap \mathbf{R} = \emptyset = \Pi \cap \mathfrak{F}\,,$$
that is points are neither regions nor abstract figures.
- Lines and planes, similarly as in classical geometry, are Cantorian sets of points: $\mathfrak{L} \cup \mathfrak{P} \subseteq \mathfrak{F}$.

From type-theoretical point of view

- In point-based geometries $\mathfrak{F}$ has the type (ι) in a hierarchy of types over the base set.
- In region based approach it has either the type $((\iota))$ or $(((\iota)))$.
- Lines and planes (and figures in general) now have either the type $(((\iota)))$ or $((((\iota))))$
- Betweenness relation if either of the type $(((\iota)),((\iota)),((\iota)))$ or $((((\iota))),(((\iota))),(((\iota))))$.
- Similarly for equidistance but with four arguments.

Summary

(i) $\mathbf{s} \neq \Pi$;

(ii) $\mathbf{s} \in \mathbf{R}$ and $\mathbf{s} \notin \mathfrak{F}$ (the space is one of regions and is not an «abstract» figure, that is it is not a Cantorian set of points);

(iii) $x \in \mathbf{R}$ and $x \neq \mathbf{s}$ iff $x \sqsubset \mathbf{s}$ (every region which is different from the space is its part and conversely, every part of the space is a region);

(iv) $\Pi \subseteq \mathcal{P}(\mathbf{R})$ or $\Pi \subseteq \mathcal{P}(\mathcal{P}(\mathbf{R}))$ and $\mathfrak{L}, \mathfrak{P} \subseteq \mathfrak{F}$ (all points are sets whose elements are regions or sets of regions; all lines and planes are abstract figures, but they are not parts of $\mathbf{s}$).

In light of the above remarks we can say that the conditions (iii)–(iv) are natural assumptions of region-based geometry.

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2. (ii) $\mathbf{s} \in \mathbf{R}$ and $\mathbf{s} \notin \mathfrak{F}$ (the space is one of regions and is not an «abstract» figure, that is it is not a Cantorian set of points);
3. (iii) $x \in \mathbf{R}$ and $x \neq \mathbf{s}$ iff $x \sqsubset \mathbf{s}$ (every region which is different from the space is its part and conversely, every part of the space is a region);
4. (iv) $\Pi \subseteq \mathcal{P}(\mathbf{R})$ or $\Pi \subseteq \mathcal{P}(\mathcal{P}(\mathbf{R}))$ and $\mathfrak{L}, \mathfrak{P} \subseteq \mathfrak{F}$ (all points are sets whose elements are regions or sets of regions; all lines and planes are abstract figures, but they are not parts of $\mathbf{s}$).

In light of the above remarks we can say that the conditions (iii)–(iv) are natural assumptions of region-based geometry.

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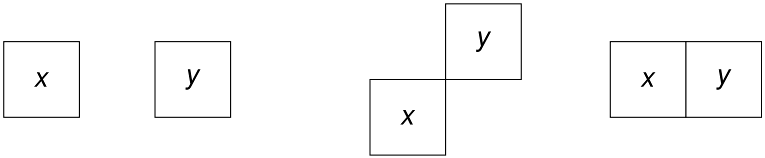
Outline

Connections structures – intuitions



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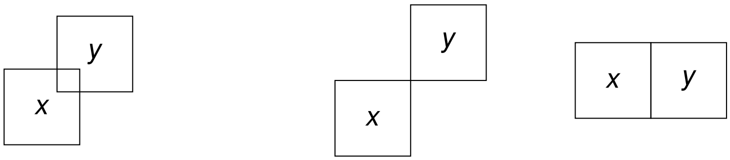

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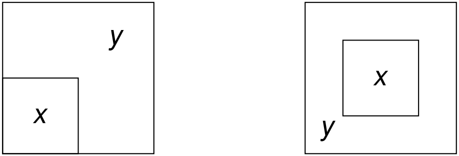


Axioms for connection

Let $\sqsubseteq$ be part of relation and **C** connection relation.

$$x \sqsubseteq y \longrightarrow x \mathbf{C} y \tag{C1}$$

$$x \mathbf{C} y \longrightarrow y \mathbf{C} x \tag{C2}$$

$$x \mathbf{C} y \wedge x \sqsubseteq z \longrightarrow z \mathbf{C} y \tag{C3}$$

$$w \text{ is the fusion of } y \text{ and } z \longrightarrow (x \mathbf{C} w \longrightarrow x \mathbf{C} y \vee x \mathbf{C} z) \tag{C4}$$

Definitions via parthood and connection

$$x \mathrel{)(} y \stackrel{\text{df}}{\longleftrightarrow} \neg x \mathbf{C} y \qquad (\mathtt{df}\,)()$$

$$x \ll y \stackrel{\text{df}}{\longleftrightarrow} y = \mathbf{1} \vee (y \neq \mathbf{1} \wedge x \mathrel{)(} -y) \qquad (\mathtt{df} \ll)$$

$$X \infty Y \stackrel{\text{df}}{\longleftrightarrow} \forall_{x \in X} \forall_{y \in Y}\, x \infty y \qquad (\mathtt{df} \infty)$$

Region-based topologies

- A. Whitehead, J. Nicod, B. Russell, T. de Laguna
- Stone spaces – **C** coincides with the overlap
- Grzegorczyk's system
- Roeper's system – limited region

Outline

Distinguished sets of regions

Idealize if you want geometry!

- half-planes: A. Śniatycki
- segments: M. Haemmerli and A. Varzi
- convex regions: I. Pratt-Hartmann, A. Cohn
- ovals: G. Gerla and R. Gruszczyński
- spheres: A. Tarski

Śniatycki's geometry

Theory of structures $\langle \mathbf{R}, \sqsubseteq, \mathbf{H} \rangle$ in which:

- $\mathbf{R}$ is a non-empty set whose elements are called regions,
- $\langle \mathbf{R}, \sqsubseteq \rangle$ is a non-atomic complete mereology,
- $\mathbf{H} \subseteq \mathbf{R}$ is a set whose elements are called half-planes (we assume that $\mathbf{1}$ is not a half-plane).

Śniatycki's geometry – axioms

$$h \in \mathbf{H} \longrightarrow -h \in \mathbf{H} \tag{H1}$$

$$\begin{aligned}\forall_{x_1,x_2,x_3\in\mathbf{R}}\Big(&\exists_{h\in\mathbf{H}}\forall_{i\in\{1,2,3\}}(x_i \bigcirc h \wedge x_i \bigcirc -h) \vee \\ &\exists_{h_1,h_2,h_3\in\mathbf{H}}(x_1 \sqsubseteq h_1 \wedge x_2 \sqsubseteq h_2 \wedge x_3 \sqsubseteq h_3 \wedge \\ &x_1 + x_2 \perp h_2 \wedge x_1 + x_3 \perp h_2 \wedge x_2 + x_3 \perp h_1)\Big)\end{aligned} \tag{H2}$$

$$\forall_{h_1,h_2,h_3\in\mathbf{H}}(h_2 \sqsubseteq h_1 \wedge h_3 \sqsubseteq h_1 \longrightarrow h_2 \sqsubseteq h_3 \vee h_3 \sqsubseteq h_2) \tag{H3}$$

$$\begin{aligned}&h_1 \cdot h_2 \sqsubseteq (h_3 \cdot h_4) + (-h_3 \cdot -h_4) \longrightarrow \\ &\quad h_3 = h_4 \vee h_1 \cdot h_2 \sqsubseteq h_3 \cdot h_4 \vee h_1 \cdot h_2 \sqsubseteq -h_3 \cdot -h_4\end{aligned} \tag{H4}$$

Points and affine geometry

- points can be defined starting from non-parallel half-planes
- affine geometry – it is what remains of Euclidean geometry when the metric notion is abandoned
  - geometry of betweenness relation
  - study of parallel lines
  - Playfair's axiom

Sphere structures

Tarski presents structures of the form $\langle \mathbf{R}, \mathbf{B}, \sqsubseteq \rangle$ such that

P0
- (i) $\langle \mathbf{R}, \sqsubseteq \rangle$ is a complete mereology,
- (ii) $\mathbf{B} \subseteq \mathbf{R}$.

- Elements of **R** are called regions.
- Elements of **B** are called mereological spheres (or simply spheres in case it follows from the context that we refer to elements of **B**).
- The notions of region, sphere and being part of are primitive notions of the analyzed theory.
- Observe that spheres are more «idealized» than, for example ovals or half-planes.

Definition of a point

Definition

By a point we mean the set of all those sphere that are concentric with a given sphere:

$$\beta \in \Pi \overset{\mathrm{df}}{\longleftrightarrow} \exists_{b \in \mathbf{B}}\, \beta = \{x \in \mathbf{B} \mid x \circledcirc b\}\,. \tag{df Π}$$

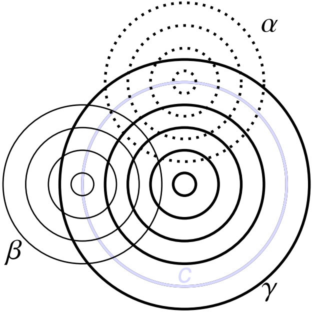
Definition

Points α and β are equidistant from a point γ iff

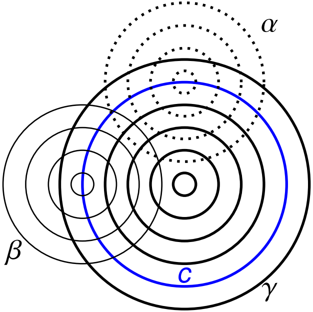
(i) $\alpha = \beta = \gamma$ or
(ii) there exists a sphere in γ such that no sphere from α or β either is part of or is exterior to this sphere.

Equidistance relation among points

Equidistance relation among points



Equidistance relation among points



Main axiom of Tarski's geometry

The first one of the specific axioms of the geometry of solids states that:

- points defined as sets of concentric spheres are points of an ordinary point-based geometry,
- the relation Δ is an ordinary equidistance relation.

$\langle \Pi, \Delta \rangle$ is a Pieri structure.

Outline

Back to idealization and abstraction

- The notion of point is mathematical interpretation of the notion of the most precise location in the perspective space.
- There are two paths leading to points (and other mathematical objects):
  - the path of idealization
  - the path of abstraction
  - those paths intertwine

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The End